Realistic Picture of 2D Harmonic Oscillator Coherent States

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We show that a 2D harmonic oscillator coherent state is a soliton which has the same evolution as a spinning top: the center of mass follows a classical trajectory and the particle rotates around its center of mass in the same direction as its spin with the harmonic oscillator frequency.

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I. INTRODUCTION

The harmonic oscillator coherent states introduced in 1926 by Schrödinger [1] have become very important in quantum optics due to Glauber [2] in 1965, who deduced the quantum theory of optical coherence from these coherent states. Note its three most important properties: 1. it describes a nondispersive wave packet, 2. its center follows a classical trajectory, 3. the Heisenberg inequalities are equalities.

We show in this paper that a coherent state of 2D harmonic oscillator can be considered as a solid which rotates around its center of mass in the same direction as its spin with an angular velocity ω, while the center of mass follows the classical trajectory of a harmonic oscillator of frequency ω.

The proof presented in this article will use a new definition of the Schrödinger current that shall be recalled here.

II. THE SCHRÖDINGER CURRENT

In 1928 Gordon [4] showed that the Dirac current can be subdivided into a convection current and a spin-dependent current. In the Pauli non relativistic approximation, this spin-dependent current can be written [4]:

$$\mathbf{J}_{Pauli-spin} = \frac{\hbar}{2m_e}\nabla\times(\Psi^*\sigma\Psi). \tag{1}$$

Let's suppose now that the particle is in a spin eigenstate, i.e. that the Pauli spinor can be written $\Psi(\mathbf{r},t) = \psi(\mathbf{r},t)\chi$ where χ is a constant spinor such as $\chi^*\chi = 1$. Holland [5] shows that the Pauli spin-dependent current becomes the Schrödinger spin-dependent current

$$\mathbf{J}_{Sch-spin} = \frac{\hbar}{2m_e}\nabla\rho\times\mathbf{u} \tag{2}$$

where $\psi = \sqrt{\rho}e^{i\frac{S}{\hbar}}$ and where $\mathbf{u} = \chi^*\sigma\chi$ is the spin vector. So that, to obtain a good approximation of the Dirac current, it is necessary to add to the classical convection current $\mathbf{J_{Sch-conv}} = \frac{i\hbar}{2m_e}(\psi\nabla\psi^* - \psi^*\nabla\psi) = \rho\frac{\nabla S}{m_e}$ the spin-dependent current (2). Then the new Schrödinger current becomes [5]:

$$\mathbf{J}_{Sch} = \rho\frac{\nabla S}{m_e} + \frac{\hbar}{2m_e}\nabla\rho\times\mathbf{u}. \tag{3}$$

It is possible to verify from the ground state of the hydrogen atom that this new definition is necessary. The classical Schrödinger convection current of the eigenfunction ψ_{100} (ground state) is zero because ψ_{100} is real. Therefore, in the ground state of the hydrogen atom $1s_{\frac{1}{2}}$, the Dirac current is equal to [6, 7]

$$\mathbf{J}_{Dirac1s_{\frac{1}{2}}} = \rho\alpha c\sin\theta\mathbf{u}_\varphi. \tag{4}$$

This is exactly equal to the value given by the Schrödinger spin-dependent current (2).

Finally, de Struyve, De Baere, De Neve and De Weirdt [8] have proved that the formula (3) is also necessary for the bosons of spin 1.

III. REALISTIC PICTURE OF THE 2D HARMONIC OSCILLATOR COHERENT STATES

In the case of the 2D harmonic oscillator, $V(\mathbf{x}) = \frac{1}{2}m\omega^2(x^2+y^2)$, the coherent states are build [3] on the initial wave function $\Psi_0(\mathbf{x}) = \left(2\pi\sigma_\hbar^2\right)^{-\frac{1}{2}} e^{-\frac{(\mathbf{x}-\xi_0)^2}{4\sigma_\hbar^2}+i\frac{m\mathbf{v}_0\cdot\mathbf{x}}{\hbar}}$ where ξ_0 and $\mathbf{v}_0$ are independent data from $\hbar$ and where $\sigma_\hbar = \sqrt{\frac{\hbar}{2m\omega}}$.

The wave function $\Psi(\mathbf{x},t)$, solution of the Schrödinger equation is then the coherent state [3]:

$$\Psi(\mathbf{x},t) = \left(2\pi\sigma_\hbar^2\right)^{-\frac{1}{2}} e^{-\frac{(\mathbf{x}-\xi(t))^2}{4\sigma_\hbar^2}+i\frac{m\mathbf{v}(t).\mathbf{x}-g(t)}{\hbar}} \tag{5}$$

where $\xi(t) = \xi_0\cos(\omega t) + \frac{\mathbf{v}_0}{\omega}\sin(\omega t)$ and $\mathbf{v}(t) = \mathbf{v}_0\cos(\omega t) - \xi_0\omega\sin(\omega t)$ respectively correspond to position and velocity of a classical particle in a potential $V(\mathbf{x}) = \frac{1}{2}m\omega^2(x^2+y^2)$: ξ_0 and $\mathbf{v}_0$ are the initial position and velocity and $g(t) = \int_0^t(\hbar\omega + \frac{1}{2}m\mathbf{v}^2(s) - \frac{1}{2}m\omega^2\xi^2(s))ds$.

The energy $E(\mathbf{x},t)$ defined by equation $i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{x},t) = E(\mathbf{x},t)\Psi(\mathbf{x},t)$ is equal to $g'(t) - m\frac{d\mathbf{v}(t)}{dt}.\mathbf{x} + i\hbar\frac{\mathbf{x}-\xi(t)}{2\sigma_\hbar^2}.\mathbf{v}(t)$.

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On the trajectory $\xi(t)$, the energy is constant and equal to

$$E(\xi(t),t) = \hbar\omega + (\frac{1}{2}m\mathbf{v}^2(t) + \frac{1}{2}m\omega^2\xi^2(t)) \\ = \hbar\omega + (\frac{1}{2}m\mathbf{v}_0^2 + \frac{1}{2}m\omega^2\xi_0^2). \quad (6)$$

Then, the coherent state is a state with a constant energy on the trajectory $\xi(t)$.

In the Schrödinger approximation (constant spin orientation), the velocity field $\mathbf{v}(\mathbf{x},t)$ of the 2D harmonic oscillator is equal to (cf. (3)):

$$\mathbf{v}(\mathbf{x},t) = \frac{1}{\rho}\mathbf{J}_{Sch}(\mathbf{x},t) = \frac{\nabla S(\mathbf{x},t)}{m} + \frac{\hbar\nabla\rho(\mathbf{x},t)}{2m\rho(\mathbf{x},t)} \times \mathbf{k} \\ = \mathbf{v}(t) + \Omega \times (\mathbf{x} - \xi(t)) \quad (7)$$

where $\mathbf{k}$ is the spin vector and where $\Omega = \omega\mathbf{k}$.

$\mathbf{v}(\mathbf{x},t)$ can be interpreted as the velocity of a solid. Its center of mass follows a classical trajectory. The solid rotates around its center of mass in the same direction as its spin with an angular velocity ω.

We have found for the coherent states of the 2D harmonic oscillator a classical geometric picture. We verify that this picture corresponds to a spread particle which satisfies the Heisenberg equalities

$$\Delta x.\Delta p_x = \frac{\hbar}{2}; \quad \Delta\ y.\Delta p_y = \frac{\hbar}{2} \quad (8)$$

and has the energy (6). Indeed, we have:

$$(\Delta x)^2 = <(x - \xi_x(t))^2> \\ = \int (x - \xi_x(t))^2 \left(2\pi\sigma_\hbar^2\right)^{-\frac{1}{2}} e^{-\frac{(x-\xi_x(t))^2}{4\sigma_\hbar^2}} dx = \sigma_\hbar^2,$$

$$(\Delta p_x)^2 = <(m\mathbf{v}_x(\mathbf{x},t) - m\mathbf{v}_x(t))^2> \\ = <m^2\omega^2(y - \xi_y(t))^2> = m^2\omega^2\sigma_\hbar^2,$$

and

$$E = <\frac{1}{2}m\mathbf{v}^2(\mathbf{x},t) + \frac{1}{2}m\omega^2\mathbf{x}^2> \\ = <\frac{1}{2}m\mathbf{v}^2(t)> + <\frac{1}{2}m\omega^2(\mathbf{x} - \xi(t))^2> \\ + <\frac{1}{2}m\omega^2\xi(t)^2> + <\frac{1}{2}m\omega^2(\mathbf{x} - \xi(t))^2> \\ = \frac{1}{2}m\mathbf{v}_0^2 + \frac{1}{2}m\omega^2\xi_0^2 + \hbar\omega.$$

Then, it is natural to assume, in this case, that the wave function represents a spread particle and its square the particle density.

The 2D harmonic oscillator ground state corresponds to $\xi_0 = 0$ and $\mathbf{v}_0 = 0$. It can be represented by a disk of density $\rho(x,y) = \left(2\pi\sigma_\hbar^2\right)^{-1} e^{-\frac{x^2+y^2}{2\sigma_\hbar^2}}$ which spins with an angular velocity ω.

IV. CONCLUSION

We have proposed for the 2D oscillator harmonic wavefunction the picture of a spread particle (soliton). However, we have still not found such a simple picture for the 3D harmonic oscillator and the hydrogen atom.

The hydrogen atom ground state, in the Schrödinger approximation, is a coherent state as the harmonic oscillator ground state: the wave packet center is also immobile ($\xi(t) = 0$) and the velocity is equal to $\mathbf{v}^\hbar(\mathbf{x},t) = \alpha c \sin\theta \mathbf{u}_\varphi = \Omega \times \frac{\mathbf{r}}{r}$ with $\Omega = \alpha c\mathbf{k}$.

Therefore, the picture of a spread particle of density $\rho(r) = \frac{1}{\pi a_0^3}\ e^{-2\frac{r}{a_0}}$, where each point (r,θ) rotates around the spin vector with a constant velocity $\alpha c \sin\theta$, satisfies Heisenberg equalities $\Delta r \Delta p = \sqrt{2}\hbar$ but not the energy $E = -\frac{1}{2}m\alpha^2c^2$.

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